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Journal of **A**dvanced **R**esearch in
Name

Vol. 00 No. 0, pp. 1-21
February 2010

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Abstract. We study a problem describing the motion of an incompressible, non-isothermal and non-Newtonian fluid, taking into account the heat convection term. The novelty here is that fluid viscosity depends on the temperature, the velocity of the fluid, and also of the deformation tensor, but not explicitly. The boundary conditions take into account the slip phenomenon on a part of the boundary of the domain. By using the notion of pseudo-monotone operators and fixed point Theorem we prove an existence result of its weak solution.

Keywords: Heat convection; Non-Newtonian fluid; Non-isothermal fluid; Tresca fluid-solid conditions; Pseudo-monotone operators; Schauder point fixed Theorem.

Mathematics Subject Classification 2010: 76A05, 76D50, 35Q35.

1 Introduction

Let ω be fixed bounded domain of $\mathbb{R}^2$, with Lipschitz continuous boundary. We suppose that ω is the bottom of the fluid domain Ω, the upper surface Γ_1 is defined by the equation $x_3 = h(x')$ where $x' = (x_1, x_2)$ and h is a positive smooth and bounded function. Then

$$\Omega = \left\{ \left(x', x_3\right) \in \mathbb{R}^3; \quad x' \in \omega \quad and \quad 0 < x_3 < h\left(x'\right) \right\}.$$

The boundary $\partial\Omega$ is composed of three parts; $\partial\Omega = \overline{\omega} \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_L$, where Γ_L is the lateral boundary. We consider a stationary problem, in the bounded domain Ω, describing the

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*Received: 3 october 2009

motion of an incompressible non-isothermal and non-Newtonian fluid. This problem is deduced (see [5]) from the three conservation laws of, mass, momentum and energy, (see for example [15,19]), where the density is assumed to be constant and equal to 1, so the mass conservation law becomes the incompressibility condition of the fluid

$$div\,(v) = 0 \quad in \quad \Omega, \tag{1.1}$$

where v is the velocity of the fluid. Many fluid flows (molten polymers in solution, oils, sludge ...) do not verify Newton's law $\sigma(v) = -\pi I + 2\mu D(v)$, with $\mu = const.$, but a more complex in which the viscosity μ varies with the strain tensor $D(v)$, the temperature θ, or also the second invariant $D_{11} = \frac{1}{2}D(v) : D(v)$. We consider here that the heat conduction phenomenon is described by Fourier's law, relating the heat flux to the temperature θ, so the energy law leads to

$$v.\nabla\theta = 2\mu\,(\theta, v, |D\,(v)|)\,D\,(v) : D\,(v) + div(K\nabla\theta) + r\,(\theta) \quad in \quad \Omega, \tag{1.2}$$

where K is a positive function defined on Ω and r is real function.

The motion of the fluid is assumed to be slow, then the momentum law leads to

$$-2div\,(\mu\,(\theta, v, |D(v)|)\,D(v)) + \nabla\pi = f \quad in \quad \Omega, \tag{1.3}$$

where π is the pressure of the fluid, f is a given vector and will be specified later.

Compared to the works [2]- [9], and to the earlier works, to our knowledge, the novelty in this study is firstly that we take into account the effects of the heat convection expressed by the presence of the left term in (1.2), secondly we consider the viscosity μ of the fluid as a function depending on its temperature, its velocity and its strain tensor. This general choice of the viscosity allows us to include the cases, of power law [13,16], Carreau law [4,20] or Bingham law [15].

This choice also allows to choose the appropriate viscosity that meets industrial applications such that, for example, the manufacture of flak vest, containing a fluid that has the ability to focus on the impact of the projectile upon contact with the flak vest.

See an other situation in [12] where the paper concerns longtime and large-data existence results for a generalization of the Navier-Stokes fluid whose viscosity depends on the shear rate and the pressure in the form $\nu = \nu(p, |D(v)|^2)$. We can see also [11] in which the authors considered flows of fluids in unbounded domains with viscosity having the same form as in [12].

In order to close the system, it remains to describe the boundary conditions for the temperature θ and the velocity v. We consider the following Neumann boundary condition

$$K\frac{\partial\theta}{\partial n} = \theta_\omega, \quad on \quad \omega, \tag{1.4}$$

where $n = (n_1, n_2, n_3)$ is the outward unit normal to $\partial\Omega$, and θ_ω is a given fixed temperature flux on ω, and the following Dirichlet condition

$$\theta = 0, \quad on \quad \Gamma_1 \cup \Gamma_L. \tag{1.5}$$

For the boundary conditions for the velocity v, let $g = (g_1, g_2, g_3)$ be a function such that

$$\int_{\partial\Omega} g.nds = 0, \quad g_3 = 0 \quad on \quad \Gamma_L, \quad g = 0 \quad on \quad \Gamma_1, \quad g.n = 0, \quad on \quad \omega,$$

the velocity on Γ_L is known and parallel to the plane (x_1, x_2) so,

$$v = g \quad on \quad \Gamma_L, \tag{1.6}$$

the upper surface Γ_1 being assumed to be fixed so

$$v = 0 \quad on \quad \Gamma_1, \tag{1.7}$$

we assume that there is no flux through ω, so the normal component of the velocity vanishes,

$$v.n = 0 \quad on \quad \omega, \tag{1.8}$$

but the tangential component v_t of the velocity is unknown and satisfies the Tresca law [2], [15] Chap.3,

$$\begin{cases} |\sigma_t| = k \Rightarrow \exists\lambda \geq 0; v_t = s - \lambda\sigma_t, \\ |\sigma_t| < k \Rightarrow v_t = s. \end{cases} \tag{1.9}$$

Where k is the upper limit for the stress, s is the velocity of the surface ω and σ_t is the tangential component of σn, where $\sigma = (\sigma_{ij})_{1\leq i,j\leq 3}$ is the stress tensor defined by

$$\sigma_{ij} = 2\mu\left(\theta, v, |D\left(v\right)|\right) d_{ij}\left(v\right) - \pi\delta_{ij}, \tag{1.10}$$

with δ_{ij} is the Kronecker symbol and $D(v)$ is the deformation tensor given by

$$D(v) = (d_{ij}(v))_{1\leq i,j\leq 3}, \quad d_{ij}(v) = \frac{1}{2}(\partial_j v_i + \partial_i v_j). \tag{1.11}$$

Here and below, we write ∂_i to design $\frac{\partial}{\partial x_i}$ and use the convention of implicit summation over repeated indices.

The term $|D(v)|$ denotes the euclidian norm of $D(v)$, that is $|D(v)|^2 = d_{ij}(v)d_{ij}(v)$, induced by the product $D(u) : D(v) = d_{ij}(u)d_{ij}(v)$.

The plan of this paper is as follows. In Section 2 we establish the variational formulation of the considered problem (1.1)-(1.9). Note that the terms $\mu\left(\theta, v, |D\left(v\right)|\right) D\left(v\right) :$

$D(v)$ and $v.\nabla\theta$ in (1.2) and $div\left(\mu\left(\theta, v, |D(v)|\right) D(v)\right)$ in (1.3) lead us to consider particular sets of the Sobolev spaces $\left(W^{1,p}(\Omega)\right)^3$ and $\left(W^{1,q}(\Omega)\right)^3$ where $p > 3$ and q its exponent conjugate, for details see the proof of Proposition 2.1. In the same section we give some lemmas needed for the next one to prove the boundedness and the coerciveness of the operator A defined by

$$\langle A(v), \varphi\rangle = \int_\Omega 2\mu\left(\theta, v, |D(v)|\right) d_{ij}(v)\partial_j\varphi_i dx.$$

In fact its coerciveness also follows from the assertion

$$\lim_{\|v\|_{1,2}\to+\infty} \frac{\|v\|_{1,2}}{\|v\|_{1,p}} \neq 0,$$

which we prove in the present paper, for notations, see the next section. Also note the fact that the function μ does not depend explicitly on its arguments leads us to assume it monotone in $|D(v)|$. This enable us to establish the monotonicity of the operator A.

In Section 3 we study the existence results of the problem 2.1, in the following three subsections.

In Subsection 3.1, for given temperature $\theta \in W^{1,q}_{\Gamma_1\cup\Gamma_L}(\Omega)$ and $f \in (W^{1,p}(\Omega))^3$, $0 \le k \in L^p(\omega)$, we prove in Theorem 3.1, using Korn inequalities and classical results of monotone operators, that there exists at least one $v_\theta \in W^{1,p}_{div}(\Omega)$ solution of the intermediate problem 3.1. We prove also in Lemma 3.2 that v_θ remains bounded in $W^{1,p}_{div}(\Omega)$, independently of the temperature θ. This is needed in Theorem 3.2 where we derive the existence of the pressure $\pi \in L^p_0(\Omega)$ satisfying the varitional inequality (2.1).

In Subsection 3.2, We consider a second intermediate problem that, for given velocity $v \in V^p_{div}$ find the temperature solution of Problem 3.2. We remark here that from the weak formulation, the temperature θ must be obtained in the space $W^{1,q}(\Omega)$. But this weak formulation led us, after linearization of the corresponding equation, to a bilinear form $B(\theta, \psi)$ defined on the space $W^{1,q}(\Omega) \times W^{1,p}(\Omega)$. And in order to apply Lax-Milgram Theorem, we have considered the Hilbert space $H^1(\Omega) \times H^1(\Omega)$, and we establish in Theorem 3.3 the existence and uniqueness of θ in $H^1(\Omega)$, solution of the linearized problem 3.3, hence $\theta \in W^{1,q}(\Omega)$, because $1 < q < 2$, see other reasons for this choice of the space $H^1(\Omega)$ in the proof of Theorem 3.3.

Using Schauder Fixed Point Theorem, we establish first in Theorem 3.4, the existence of $\theta \in H^1_{\Gamma_1\cup\Gamma_L}(\Omega)$ solution of the second intermediate problem (3.2), then we prove its uniqueness in Theorem 3.5 by a monotony method.

In Subsection 3.3, We recapitulate the necessary assumptions used to prove the existence of at least one solution of the variational global problem 2.1, and establish it in Theorem 3.6.

2 Weak formulation

Let p be a real number such that $p > 1$, and q its conjugate exponent, that is $\frac{1}{p}+\frac{1}{q}=1$. We know from Lemma 2.2 in [17] that for $g \in \left(W^{1-\frac{1}{p},p}(\partial\Omega)\right)^3$, there exists a function G in $\left(W^{1,p}(\Omega)\right)^3$ such that,

$$div(G) = 0 \quad in \quad \Omega, \quad G = g \quad on \quad \Gamma_L \cup \Gamma_1, \quad G.n = 0 \quad on \quad \omega.$$

To establish a weak formulation of the problem, we introduce the following sets,

$$V^p = \left\{\varphi \in \left(W^{1,p}(\Omega)\right)^3; \quad \varphi = G \quad on \quad \Gamma_L, \quad \varphi = 0 \quad on \quad \Gamma_1 \quad and \quad \varphi.n = 0 \quad on \quad \omega\right\},$$

$$V^p_{div} = \{\varphi \in V^p; \quad div(\varphi) = 0 \quad in \quad \Omega\},$$

V^p and V^p_{div} are convex closed subsets of $\left(W^{1,p}(\Omega)\right)^3$. We also define the spaces,

$$V^p_0 = \left\{\varphi \in \left(W^{1,p}(\Omega)\right)^3; \quad \varphi = 0, \quad on \quad \Gamma_L \cup \Gamma_1 \quad and \quad \varphi.n = 0 \quad on \quad \omega\right\}$$

$$V^p_{0,div} = \{\varphi \in V^p_0; \quad div(\varphi) = 0 \quad in \quad \Omega\},$$

$$L^p_0(\Omega) = \left\{u \in L^p(\Omega); \int_\Omega u(x)dx = 0\right\},$$

and we denote,

$$V^p_\Gamma = \left\{\varphi \in \left(W^{1,p}(\Omega)\right)^3; \quad \varphi = 0 \quad on \quad \Gamma\right\},$$

$$W^{1,p}_\Gamma(\Omega) = \left\{\varphi \in W^{1,p}(\Omega); \quad \varphi = 0 \quad on \quad \Gamma\right\},$$

where Γ is a subset of $\partial\Omega$ with $|\Gamma| := meas(\Gamma) \neq 0$.
Remark that $V^p_\Gamma = \left(W^{1,p}_\Gamma(\Omega)\right)^3$ and $V^p, V^p_0 \subset V^p_{\Gamma_1}$. We denote, as usual the norm of the Lebesgue space $L^p(\Omega)$ by $\|v\|_p = \left(\int_\Omega |v|^p\,dx\right)^{\frac{1}{p}}$, and the norms of the Banach spaces V^p_Γ and $W^{1,p}_\Gamma(\Omega)$ are designed equally by $\|v\|_{1,p} = \left(\int_\Omega |\nabla v|^p\,dx\right)^{\frac{1}{p}}$.

In the following proposition, we deduce rigorously the variational formulation of Problem (1.1)-(1.9). Also note that we only give the necessary assumptions on the data of each part of the paper. We summarize, the necessary hypotheses needed for the final result in Subsection 3.3.

Proposition 2.1. Let $p > 3$ and q its conjugate exponent. For $f \in (W^{1,p}(\Omega))^3$, $0 \leq k \in L^p(\omega)$, $\mu \in L^\infty\left(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}_+\right)$, $K \in L^\infty(\Omega)$ and $r \in L^\infty(\mathbb{R})$, the weak formulation of the strong problem (1.1)-(1.9) leads to the following variational problem,

Problem 2.1. Find $v \in V^p_{div}$, $\pi \in L^p_0(\Omega)$ and $\theta \in W^{1,q}_{\Gamma_1 \cup \Gamma_L}(\Omega)$ such that

$$a(\theta, v, \varphi - v) - (\pi, div(\varphi)) + j(\varphi) - j(v) \geq (f, \varphi - v) \quad \forall \varphi \in V^q, \tag{2.1}$$

$$B(\theta, \psi) = L(\theta, \psi) \quad \forall \psi \in W^{1,p}_{\Gamma_1 \cup \Gamma_L}(\Omega), \tag{2.2}$$

where

$$a(\theta, v, \varphi - v) = \int_\Omega 2\mu(\theta, v, |D(v)|) d_{ij}(v) \partial_j (\varphi_i - v_i)\, dx,$$

$$L(\theta, \psi) = 2\int_\Omega \mu(\theta, v, |D(v)|) |D(v)|^2 \psi dx + \int_\Omega r(\theta) \psi dx + \int_\omega \theta_\omega \psi dx', \tag{2.3}$$

and

$$B(\theta, \psi) = \int_\Omega K(x) \nabla\theta . \nabla\psi dx + \int_\Omega \psi v_i \partial_i \theta dx, \quad j(\varphi) = \int_\omega k\, |\varphi - s|\, dx'. \tag{2.4}$$

Proof. Observe first that since $p > 3$ and Ω is bounded, if $v = (v_1, v_2, v_3) \in V^p$ then $v_i \in L^\infty(\Omega)$. Hence for $\psi \in W^{1,p}_{\Gamma_1 \cup \Gamma_L}(\Omega)$ and $\theta \in W^{1,q}_{\Gamma_1 \cup \Gamma_L}(\Omega)$ the second integral in $B(\theta, \psi)$ makes sense. For the first integral in $L(\theta, \psi)$ we have $|D(v)|^2 \in L^{\frac{p}{2}}(\Omega)$ then $\mu(\theta, v, |D(v)|) |D(v)|^2 \psi \in L^1(\Omega)$ until μ is bounded. Also $a(\theta, v, \varphi - v)$ is well defined since $d_{ij}(v) \in L^p(\Omega)$ and $\partial_j(\varphi_i - v_i) \in L^q(\Omega) - L^p(\Omega) \subset L^q(\Omega)$, because $1 < q < p$. To obtain the variational inequality (2.1), we have from (1.3) and (1.10)

$$-\partial_j \sigma_{ij} = f_i, \quad i = 1, 2, 3. \tag{2.5}$$

Multiplying (2.5) by $\varphi_i - v_i$ where $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in V^q$ and integrating over Ω, we obtain

$$\int_\Omega \sigma_{ij} \partial_j (\varphi_i - v_i)\, dx - \int_{\partial\Omega} \sigma_{ij} (\varphi_i - v_i)\, n_j ds = \int_\Omega f_i (\varphi_i - v_i)\, dx. \tag{2.6}$$

Since $\varphi_i - v_i = 0$ on $\Gamma_1 \cup \Gamma_L$,

$$\int_{\partial\Omega} \sigma_{ij} (\varphi_i - v_i)\, n_j ds = \int_\omega \sigma_{ij} (\varphi_i - v_i)\, n_j dx'.$$

Remark that $\sigma_{ij} n_j$ is the i-th component of the vector σn, which can be written on the form $\sigma n = \sigma_t + \sigma_n n$, with $\sigma_t = (\sigma_{t_1}, \sigma_{t_2}, \sigma_{t_3})$ and $\sigma_n = \sigma n.n$, from which we have $\sigma_{ij} n_j = \sigma_{t_i} + \sigma_n n_i$. Using this equality, we get

$$\int_\omega \sigma_{ij} (\varphi_i - v_i)\, n_j dx' = \int_\omega \sigma_{t_i} (\varphi_i - v_i)\, dx' + \int_\omega \sigma_n n_i (\varphi_i - v_i)\, dx'.$$

Knowing that from (1.8), $n_i(\varphi_i - v_i) = 0$ on ω, then

$$\int_\omega \sigma_{ij}(\varphi_i - v_i) n_j dx' = \int_\omega \sigma_{t_i}(\varphi_i - v_i) dx'$$

and (2.6) becomes

$$\int_\Omega \sigma_{ij}\partial_j(\varphi_i - v_i) dx = \int_\omega \sigma_{t_i}(\varphi_i - v_i) dx' + \int_\Omega f_i(\varphi_i - v_i) dx. \tag{2.7}$$

Let us involve the Tresca condition. For this end, we add to both sides of (2.7) the term $\int_\omega k(|\varphi - s| - |v - s|) dx'$, then

$$\int_\Omega \sigma_{ij}\partial_j(\varphi_i - v_i) dx + \int_\omega k(|\varphi - s| - |v - s|) dx' = \int_\Omega f_i(\varphi_i - v_i) dx + A, \tag{2.8}$$

with

$$A = \int_\omega [\sigma_{t_i}(\varphi_i - v_i) + k(|\varphi - s| - |v - s|)] dx'.$$

Let us prove that A is positive. First, Following [15] Chap.3 page 140, we prove that the condition (1.9) is equivalent to

$$(v_t - s).\sigma_t + k|v_t - s| = 0, \tag{2.9}$$

indeed if (1.9) holds and $|\sigma_t| = k$, then $v_t = s - \lambda\sigma_t$ for some$\lambda \geq 0$, so

$$(v_t - s).\sigma_t + k|v_t - s| = -\sigma_t.\sigma_t + k\lambda|\sigma_t| = -\lambda\sigma_t^2 + \lambda\sigma_t^2 = 0.$$

Now if $|\sigma_t| < k$, by (1.9) $v_t = s$ and (2.9) holds.
Conversely if $|\sigma_t| = k$, then by (2.9) $(v_t - s).\sigma_t = -|\sigma_t||v_t - s|$, so there exists $\lambda \geq 0$ such that $v_t - s = -\lambda\sigma_t$, that is $v_t = s - \lambda\sigma_t$. The first part of (1.9) is shown. If $|\sigma_t| < k$, then by (2.9) we have $(v_t - s).\sigma_t + k|v_t - s| = 0 \geq |v_t - s|(k - |\sigma_t|)$, thus $v_t - s = 0$ because $|\sigma_t| < k$. The second part of (1.9) holds, and the assertion is proved.
Now by (1.8) we deduce that $v = v_t$ on ω, hence

$$A = \int_\omega (\sigma_t.(\varphi - s) + k|\varphi - s|) dx'.$$

But $\sigma_t.(\varphi - s) \geq -|\sigma_t||\varphi - s|$, and since $|\sigma_t| \leq k$ on ω, it follows that

$$\sigma_t.(\varphi - s) + k|\varphi - s| \geq 0 \quad on \quad \omega,$$

this shows that A is positive. And (2.8) becomes

$$\int_\Omega \sigma_{ij}\partial_j(\varphi_i - v_i) dx + \int_\omega k(|\varphi - s| - |v - s|) dx' \geq \int_\Omega f_i(\varphi_i - v_i) dx.$$

Replacing σ_{ij} by its expression (1.10) and using (1.1), we obtain the variationnal inequality for the velocity field v. For all $\varphi \in V^q$

$$\int_\Omega 2\mu(\theta, v, |D(v)|) d_{ij}(v) \partial_j (\varphi_i - v_i)\, dx - \int_\Omega \pi div(\varphi) dx$$
$$+ \int_\omega k\left(|\varphi - s| - |v - s|\right) dx' \geq \int_\Omega f_i (\varphi_i - v_i)\, dx.$$

Similarly, by multiplying (1.2) by $\psi \in W^{1,p}_{\Gamma_1 \cup \Gamma_L}(\Omega)$ we obtain (2.2). □

Subsequently, we will use the following well known results,

$$\int_\Omega |D(u)|^2\, dx \leq \int_\Omega |\nabla u|^2\, dx, \qquad \forall u \in \left(W^{1,p}(\Omega)\right)^3, \tag{2.10}$$

$$\int_\Omega |D(u)|^2\, dx \geq \frac{1}{2} \int_\Omega |\nabla u|^2\, dx, \qquad \forall u \in V_0^p. \tag{2.11}$$

3 Existence results

We assume that the function μ defined on $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}_+$ is such that

$$\exists \mu_0, \mu_1 \in \mathbb{R}; \quad 0 < \mu_0 \leq \mu(t, u, s) \leq \mu_1, \quad \forall\, (t, u, s) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}_+, \tag{3.1}$$

$$\textit{the} \quad \textit{function} \quad s \mapsto \mu(.,.,s) \quad \textit{is} \quad \textit{monotone} \quad \textit{on} \quad \mathbb{R}_+. \tag{3.2}$$

3.1 First intermediate problem

From the variational inequality (2.1), we obtain the following intermediate problem

Problem 3.1. For given $\theta \in W^{1,q}_{\Gamma_1 \cup \Gamma_L}(\Omega)$ and $f \in (W^{1,p}(\Omega))^3$, we look for $v \in V^p_{div}$ satisfying the following variational inequality

$$a\left(\theta, v, \varphi - v\right) + j\left(\varphi\right) - j\left(v\right) \geq \left(f, \varphi - v\right), \quad \forall \varphi \in V^q_{div}. \tag{3.3}$$

To solve this problem we will use the Nonlinear Operators Theory [18]. We define the operator

$$A : V^p_{\Gamma_1} \to (V^p_{\Gamma_1})' \quad by \quad \langle A(v), \varphi \rangle = a(\theta, v, \varphi), \tag{3.4}$$

where $\langle .,. \rangle$ is the duality brackets between $(V^p_{\Gamma_1})'$ and $V^p_{\Gamma_1}$, and we denote by $\Lambda_{V_{div}}$ the characteristic function of V^p_{div},

$$\Lambda_{V_{div}}(u) = 0 \quad if \quad u \in V^p_{div}, \quad \text{and} \quad \Lambda_{V_{div}}(u) = +\infty \quad if \quad u \notin V^p_{div}.$$

Then the variational inequality (3.3) becomes,

$$\langle A(v), \varphi - v \rangle + j\left(\varphi\right) + \Lambda_{V_{div}}(\varphi) - j\left(v\right) - \Lambda_{V_{div}}(v) \geq \left(f, \varphi - v\right), \forall \varphi \in V^q_{\Gamma_1}. \tag{3.5}$$

Lemma 3.1. *The operator A defined by (3.4) is bounded, hemicontinuous and monotone on $V^p_{\Gamma_1}$.*

Proof. For any $v \in V^p_{\Gamma_1}$ and $\varphi \in V^q_{\Gamma_1}$, we have

$$|\langle A(v), \varphi\rangle| = \left| \int_\Omega 2\mu\left(\theta, v, |D(v)|\right) d_{ij}(v)\partial_j\varphi_i dx \right| \leq 2\mu_1 \left| \int_\Omega d_{ij}(v) d_{ij}(\varphi) dx \right|,$$

using Hölder's and Minkowski's inequalities, and (3.1), we get

$$\begin{aligned} |\langle A(v), \varphi\rangle| &\leq 2\mu_1 \int_\Omega \left(\sum_{i,j=1}^{3} |d_{ij}(v)|^p \right)^{\frac{1}{p}} \left(\sum_{i,j=1}^{3} |d_{ij}(\varphi)|^q \right)^{\frac{1}{q}} dx \\ &\leq 2\mu_1 \int_\Omega \left(\sum_{i,j=1}^{3} |\partial_j v_i|^p \right)^{\frac{1}{p}} \left(\sum_{i,j=1}^{3} |\partial_j \varphi_i|^q \right)^{\frac{1}{q}} dx \end{aligned}$$

so

$$|\langle A(v), \varphi\rangle| \leq 2\mu_1 \left(\int_\Omega |\nabla v|^p \, dx \right)^{\frac{1}{p}} \left(\int_\Omega |\nabla \varphi|^q \, dx \right)^{\frac{1}{q}} \leq 2\mu_1 \|v\|_{1,p} \|\varphi\|_{1,q} \quad \forall (v, \varphi) \in V^p_{\Gamma_1} \times V^q_{\Gamma_1},$$

then A is bounded.

We show that A is hemicontinuous. For this, we prove that for any $u, v, w \in V^p_{\Gamma_1}$, the function

$$\alpha : \mathbb{R} \to \mathbb{R}, \quad \alpha(t) = \langle A(u + tv), w\rangle$$

is continuous. We have

$$\alpha(t) = \int_\Omega 2\mu\left(\theta, u + tv, |D(u + tv)|\right) D(u + tv) : D(w) \, dx.$$

The function

$$t \mapsto s(t) = 2\mu\left(\theta, u + tv, |D(u + tv)|\right) D(u + tv) : D(w)$$

is obviously continuous on $\mathbb{R}$. Let (t_n) be a sequence converging to t in $\mathbb{R}$. Then $s_n := s(t_n) \in L^1(\Omega)$ and (s_n) converges to $s(t)$ when n goes to $+\infty$.
The sequence (t_n) is bounded, so there exists $M > 0$ such that $|t_n| < M, \forall n \geq 0$.
Then we have

$$|s_n| \leq g := 2\mu_1 \left(|D(u)| \, |D(w)| + M \, |D(v)| \, |D(w)| \right).$$

Since $g \in L^1(\Omega)$ and is positive, by Dominated Convergence Theorem, we deduce that $s(t) \in L^1(\Omega)$ and

$$\lim_{n \longrightarrow +\infty} \int_\Omega s_n dx = \int_\Omega s(t) dx,$$

that is

$$\lim_{n\longrightarrow+\infty}\alpha\left(t_{n}\right)=\alpha\left(t\right).$$

This shows that A is hemicontinuous. For the monotonicity of A, we establish that,

$$\left\langle A\left(u\right)-A\left(v\right),u-v\right\rangle\geq0,\quad\forall u,v\in V_{\Gamma_{1}}^{p}.$$

We have,

$$\left\langle A\left(u\right)-A\left(v\right),u-v\right\rangle=2\int_{\Omega}\left[\mu\left(\theta,u,\left|D\left(u\right)\right|\right)d_{ij}\left(u\right)-\mu\left(\theta,v,\left|D\left(v\right)\right|\right)d_{ij}\left(v\right)\right]\partial_{j}\left(u_{i}-v_{i}\right)dx$$

we have also

$$\begin{aligned}\left\langle A\left(u\right)-A\left(v\right),u-v\right\rangle &= \int_{\Omega}2\mu\left(\theta,u,\left|D\left(u\right)\right|\right)d_{ij}\left(u\right)\partial_{j}u_{i}dx+\int_{\Omega}2\mu\left(\theta,v,\left|D\left(v\right)\right|\right)d_{ij}\left(v\right)\partial_{j}v_{i}dx\\&- \int_{\Omega}2\mu\left(\theta,u,\left|D\left(u\right)\right|\right)d_{ij}\left(u\right)\partial_{j}v_{i}dx-\int_{\Omega}2\mu\left(\theta,v,\left|D\left(v\right)\right|\right)d_{ij}\left(v\right)\partial_{j}u_{i}dx.\end{aligned}$$

By the fact that

$$d_{ij}\left(u\right)\partial_{j}v_{i}=d_{ij}\left(u\right)d_{ij}\left(v\right)=D\left(u\right):D\left(v\right)\leq\left|D\left(u\right)\right|\left|D\left(v\right)\right|$$

we obtain

$$\begin{aligned}\left\langle A\left(u\right)-A\left(v\right),u-v\right\rangle &\geq \int_{\Omega}2\mu\left(\theta,u,\left|D\left(u\right)\right|\right)\left|D\left(u\right)\right|^{2}dx+\int_{\Omega}2\mu\left(\theta,v,\left|D\left(v\right)\right|\right)\left|D\left(v\right)\right|^{2}dx\\&- 2\int_{\Omega}\left(\mu\left(\theta,u,\left|D\left(u\right)\right|\right)+\mu\left(\theta,v,\left|D\left(v\right)\right|\right)\right)\left|D\left(u\right)\right|\left|D\left(v\right)\right|dx.\end{aligned}$$

Now since, $2\left|D\left(u\right)\right|\left|D\left(v\right)\right|\leq\left|D\left(u\right)\right|^{2}+\left|D\left(v\right)\right|^{2}$ we get

$$\left\langle A\left(u\right)-A\left(v\right),u-v\right\rangle\geq\int_{\Omega}\left(\mu\left(\theta,u,\left|D\left(u\right)\right|\right)-\mu\left(\theta,v,\left|D\left(v\right)\right|\right)\right)\left(\left|D\left(u\right)\right|^{2}-\left|D\left(v\right)\right|^{2}\right)dx.$$

Using (3.2), we deduce that

$$\left\langle A\left(u\right)-A\left(v\right),u-v\right\rangle\geq0\quad\forall u,v\in u,v\in V_{\Gamma_{1}}^{p},$$

That is A is monotone. □

Now we are in position to give the existence Theorem for Problem 3.1.

Theorem 3.1. *Assume that (3.1) and (3.2) are satisfied, $f\in(W^{1,p}(\Omega))^{3}$ and $0\leq k\in L^{p}(\omega)$, then for fixed θ in $W_{\Gamma_{1}\cup\Gamma_{L}}^{1,q}(\Omega)$ there exists $v_{\theta}\in V_{div}^{p}$ solution of (3.3).*

Proof. From Lemma 3.1, we conclude that the operator A is pseudo-monotone. The function $v \mapsto j(v) + \Lambda_{V_{div}}(v)$ is convex, proper and lower semi-continuous. Let us check the coercivity condition, that is

$$\exists v^* \in V^p_{\Gamma_1} \quad such \quad that \quad j(v^*) + \Lambda_{V_{div}}(v^*) < +\infty$$

and

$$\lim_{\|v\|_{1,p} \to +\infty} \frac{\langle A(v), v - v^* \rangle + j(v) + \Lambda_{V_{div}}(v)}{\|v\|_{1,p}} = +\infty.$$

We choose $v^* = G$, and observe, from their explicit expressions, that

$$\text{if} \quad \lim_{\|v\|_{1,p} \to \infty} \frac{\langle A(v-G), v-G \rangle}{\|v\|_{1,p}} = +\infty \quad \text{then} \quad \lim_{\|v\|_{1,p} \to \infty} \frac{\langle A(v), v-G \rangle}{\|v\|_{1,p}} = +\infty. \tag{3.6}$$

Indeed

$$\begin{aligned} \langle A(v-G), v-G \rangle &= 2\int_\Omega \mu(\theta, v-G, |D(v-G)|) \Big(|D(v)|^2 - 2D(v) : D(G) \Big)\, dx \\ &+ 2\int_\Omega \mu(\theta, v-G, |D(v-G)|)\, |D(G)|^2\, dx, \end{aligned}$$

and

$$\langle A(v), v-G \rangle = 2\int_\Omega \mu(\theta, v, |D(v)|) \Big(|D(v)|^2 - D(v) : D(G) \Big)\, dx.$$

Now since the function μ is positive and bounded on $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}_+$ and $\|G\|^2_{1,2}\, \|v\|^{-1}_{1,p} \to 0$ when $\|v\|_{1,p} \to \infty$, (3.6) follows. Then we can use here $\langle A(v-G), v-G \rangle$ instead of $\langle A(v), v-G \rangle$. For $v \in V^p_{\Gamma_1}$, by positivity of the function $j + \Lambda_{V_{div}}$ and (3.1) we obtain

$$\langle A(v-G), v-G \rangle + j(v) + \Lambda_{V_{div}}(v) \geq 2\mu_0 \int_\Omega |D(v-G)|^2 dx.$$

We have $v - G \in V^p_0$ then by (2.11), we get

$$\langle A(v-G), v-G \rangle + j(v) + \Lambda_{V_{div}}(v) \geq \mu_0 \|v - G\|^2_{1,2}$$

and by Cauchy-Schwarz inequality

$$\mu_0 \|v - G\|^2_{1,2} \geq \mu_0 \|v\|^2_{1,2} - 2\mu_0 \|v\|_{1,2} \|G\|_{1,2} + \mu_0 \|G\|^2_{1,2},$$

so, we obtain

$$\frac{\langle A(v-G), v-G \rangle + j(v) + \Lambda_{V_{div}}(v)}{\|v\|_{1,p}} \geq \mu_0 \left(\|v\|_{1,2} \frac{\|v\|_{1,2}}{\|v\|_{1,p}} - 2\|G\|_{1,2} \frac{\|v\|_{1,2}}{\|v\|_{1,p}} + \frac{\|G\|^2_{1,2}}{\|v\|_{1,p}} \right) \tag{3.7}$$

By the continuity of the embedding $V^p_{\Gamma_1} \subset V^2_{\Gamma_1}$, we deduce that there exists a constant $c > 0$ depending only on Ω and p such that $\|v\|_{1,2} \leq c\,\|v\|_{1,p}$. Letting $\|v\|_{1,2} \to \infty$, then $\|v\|_{1,p} \to \infty$ and we have

$$\lim_{\|v\|_{1,2} \to +\infty} \frac{\|v\|_{1,2}}{\|v\|_{1,p}} \neq 0, \tag{3.8}$$

indeed, for $v \in V^p_{\Gamma_1}$ and $v \neq 0$, set $w = v\,\|v\|^{-1}_{1,p}$ then

$$\|w\|_{1,p} = 1. \tag{3.9}$$

But

$$\lim_{\|v\|_{1,2} \to +\infty} \frac{\|v\|_{1,2}}{\|v\|_{1,p}} = 0 \Longrightarrow \lim_{\|v\|_{1,2} \to +\infty} \|w\|_{1,2} = 0.$$

This is a contradition with (3.9), so (3.8) holds. Now by letting $\|v\|_{1,2} \to \infty$, we get $\|v\|_{1,p} \to \infty$ and we can deduce, by (3.8), that the right side hand of (3.7) tends to $+\infty$. consequently

$$\frac{\langle A(v-G), v-G\rangle + j\,(v) + \Lambda_{V_{div}}(v)}{\|v\|_{1,p}} \to\ +\infty.$$

So by (3.6),

$$\frac{\langle A(v), v-G\rangle + j\,(v) + \Lambda_{V_{div}}(v)}{\|v\|_{1,p}} \to\ +\infty.$$

This shows that the coercivity condition is satisfied. Applying Theorem 8.5 chap. 2 in [18], we conclude that (3.5) and hence (3.3) admits a solution v_θ in the space V^p_{div}. $\square$

Before stating the existence Theorem of the pressure we need to prove the following lemma.

Lemma 3.2. *The solution v_θ of Problem 3.1 is bounded in V^p_{div} independently of the temperature θ.*

Proof. As v_θ satisties the following variational inequality

$$a\,(\theta, v_\theta, \varphi - v_\theta) + j\,(\varphi) - j\,(v_\theta) \geq (f, \varphi - v_\theta)\,, \quad \forall \varphi \in V^q_{div},$$

by taking $\varphi = G \in V^p_{div} \subset V^q_{div}$, we get

$$\langle A(v_\theta), v_\theta - G\rangle \leq (f, v_\theta) - (f, G) + j\,(G)\,, \tag{3.10}$$

because j is positive. Remark that we can write

$$\begin{aligned}\langle A(v_\theta), v_\theta - G\rangle \quad &= \quad 2\int_\Omega \mu(\theta, v_\theta, |D(v_\theta)|) d_{ij}(v_\theta - G)\partial_j(v^i_\theta - G_i)dx \\ &+ \quad 2\int_\Omega \mu(\theta, v_\theta, |D(v_\theta)|) d_{ij}(G)\partial_j(v^i_\theta - G_i)dx,\end{aligned}$$

where v_θ^i is the i-th component of v_θ. Then

$$\begin{aligned}\langle A(v_\theta), v_\theta - G\rangle &= 2\int_\Omega \mu(\theta, v_\theta, |D(v_\theta)|)\left(|D(v_\theta - G)|^2 + D(G):D(v_\theta)\right)dx\\ &- 2\int_\Omega \mu(\theta, v_\theta, |D(v_\theta)|)\,|D(G)|^2\,dx. \end{aligned}\tag{3.11}$$

Now from (3.1), (3.10), (3.11), (2.11), (2.10) and Hölder's inequality we get

$$\begin{aligned}\mu_0 \|v_\theta - G\|_{1,2}^2 &\le 2\mu_1 \|v_\theta\|_{1,2}\|G\|_{1,2} + \|f\|_{W^{1,p}}\|v_\theta\|_{1,q}\\ &+ 2\mu_1 \|G\|_{1,2}^2 + \|f\|_{W^{1,p}}\|G\|_{1,q} + j(G),\end{aligned}$$

hence

$$\begin{aligned}\mu_0 \|v_\theta\|_{1,2}^2 \le 2(\mu_0+\mu_1)\|v_\theta\|_{1,2}\|G\|_{1,2} &+ \|f\|_{W^{1,p}}\|v_\theta\|_{1,q} + 2\mu_1\|G\|_{1,2}^2\\ &+ \|f\|_{W^{1,p}}\|G\|_{1,q} + j(G).\end{aligned}\tag{3.12}$$

From the continuous embedding $V_{\Gamma_1}^2 \subset V_{\Gamma_1}^q$, there exists a positive constant β such that,

$$\|v_\theta\|_{1,q} \le \beta\,\|v_\theta\|_{1,2}\,.$$

Then (3.12) becomes

$$\begin{aligned}\mu_0 \|v_\theta\|_{1,2}^2 \le 2(\mu_0+\mu_1)\|v_\theta\|_{1,2}\|G\|_{1,2} &+ \beta\|f\|_{W^{1,p}}\|v_\theta\|_{1,2} + 2\mu_1\|G\|_{1,2}^2\\ &+ \|f\|_{W^{1,p}}\|G\|_{1,q} + j(G).\end{aligned}\tag{3.13}$$

By (3.13) we deduce that there exists a positive constant C independent of θ such that

$$\|v_\theta\|_{1,2} \le C. \tag{3.14}$$

Indeed, on the contrary, by dividing the two sides of (3.13) by $\|v_\theta\|_{1,2}^2$ and letting $\|v_\theta\|_{1,2} \to +\infty$ we obtain $\mu_0 \le 0$. This is a contradiction because $\mu_0 > 0$, then (3.14) holds.

By (3.8), $\|v_\theta\|_{1,2}^{-1}\|v_\theta\|_{1,p}$ is bounded for $\|v_\theta\|_{1,2}$ large enough, so it follows from (3.14), the existence of a positive constant C' independent of θ such that

$$\|v_\theta\|_{1,p} \le C'. \tag{3.15}$$

The lemma is proved. $\square$

Theorem 3.2. *Under the assumptions of* Theorem 3.1*, there exists a unique* $\pi \in L_0^p(\Omega)$ *satisfying equation (2.1).*

Proof. Let v_θ be the solution of (3.3). By taking $\varphi = v_\theta \pm \phi$, for all $\phi \in V^q_{0,div}$, we obtain from (3.3) the following variational equation,

$$a(\theta, v_\theta, \phi) = (f, \phi), \quad \forall \phi \in V^q_{0,div}. \tag{3.16}$$

Consider the linear form F defined on V^q_0 by

$$F(\phi) = a(\theta, v_\theta, \phi) - (f, \phi).$$

We prove that F is continuous on V^q_0. For all ϕ in V^q_0 we have

$$|F(\phi)| \leq 2|\Omega|^{\frac{p-2}{p}} \mu_1 \|v_\theta\|_{1,p} \|\phi\|_{1,p} + |\Omega|^{\frac{p-2}{p}} \|f\|_{1,p} \|\phi\|_{1,q},$$

by Lemma 3.2 we have (3.15) so

$$|F(\phi)| \leq |\Omega|^{\frac{p-2}{p}} \left(2\mu_1 C' + \|f\|_{1,p}\right) \|\phi\|_{1,q}.$$

This shows the continuity of F, and that $F \in W^{-1,p}(\Omega)$. And since

$$F(\phi) = 0, \quad \forall \phi \in V^q_{0,div},$$

by De Rham's Theorem in [1] page 116, we deduce the existence of a unique $\pi \in L^p_0(\Omega)$, such that

$$F(\phi) = \langle \nabla \pi, \phi \rangle \quad \forall \phi \in V^q_0,$$

using Green's formula we deduce that

$$2div\left(\mu(\theta, v_\theta, |D(v_\theta)|)\, D(v_\theta)\right) + f = \nabla \pi.$$

By multiplying this equality by $\varphi \in V^q$ and using Green's formula again, we deduce that $(v_\theta, \pi) \in V^p_{div} \times L^p_0(\Omega)$ satisfies (2.1). □

3.2 Second intermediate problem

Recall that the temperature satisfies the variational equation (2.2) with (2.3) and (2.4). We assume that the function K is also such that

$$\exists k_0, k_1 \in \mathbb{R}; \quad 0 < k_0 \leq K(x) \leq k_1, \quad \forall x \in \Omega. \tag{3.17}$$

In fact we must take $p \geq 4$ and seek the solution θ in the subspace $H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$ of $W^{1,q}_{\Gamma_1 \cup \Gamma_L}(\Omega)$, this choice will be justified below in the proof of Theorem 3.4. Let us consider the second intermediate problem.

Problem 3.2. For given $v \in V^p_{div}$, find $\theta \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$ solution of the equation,

$$B(\theta, \psi) = L(\theta, \psi), \quad \forall \psi \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega). \tag{3.18}$$

Remark that, from (2.3), L depends on v. To study this nonlinear problem we consider first the following corresponding linearized problem.

Problem 3.3. For given $v \in V_{div}^p$ and $\eta \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$, find $\theta \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$ solution of the equation,

$$B(\theta, \psi) = L(\eta, \psi), \quad \forall \psi \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega). \tag{3.19}$$

Theorem 3.3. *Assume that $p \geq 4$, (3.17) and the assumptions of* Theorem 3.1 *hold. Then for $\eta \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega), v \in V_{div}^p, \theta_\omega \in L^2(\omega)$ and $r \in L^\infty(\mathbb{R})$, there exists a unique $\theta \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$ solution for (3.19).*

Proof. The bilinear form B is continuous on $H^1_{\Gamma_1 \cup \Gamma_L}(\Omega) \times H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$. Indeed, by (3.17) and since $v \in V_{div}^p \subset (L^\infty(\Omega))^3$, there exists $M > 0$ such that

$$|B(\theta, \psi)| \leq k_1 \|\theta\|_{1,2} \|\psi\|_{1,2} + M \|\theta\|_{1,2} \|\psi\|_2 .$$

Using the Poincaré's inequality we get, for some positive constant C,

$$|B(\theta, \psi)| \leq (k_1 + MC) \|\theta\|_{1,2} \|\psi\|_{1,2} .$$

This shows the continuity of the form B. Let us prove that B is coercive. We have, for any $\theta \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$,

$$B(\theta, \theta) = \int_\Omega K(x) |\nabla \theta|^2 dx + \int_\Omega \theta v_i \partial_i \theta dx \geq k_0 \int_\Omega |\nabla \theta|^2 dx + \int_\Omega \frac{1}{2} v_i \partial_i (\theta^2) dx.$$

By Green's formula, we obtain

$$\vartheta := \int_\Omega v_i \partial_i (\theta^2) dx = \int_{\partial\Omega} \theta^2 v_i n_i ds - \int_\Omega \theta^2 \partial_i v_i dx. \tag{3.20}$$

Since $\theta = 0$ on $\Gamma_1 \cup \Gamma_L$, $n.v = 0$ on ω and $div(v) = 0$ in Ω we deduce that $\vartheta = 0$, and then

$$B(\theta, \theta) \geq k_0 \|\theta\|^2_{1,2}, \quad \forall \theta \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega).$$

The coercivity of B follows. We prove that the linear form $L(\eta, .)$ is continuous on $H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$. Let $\psi \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$, by the continuous embedding $H^1_{\Gamma_1 \cup \Gamma_L}(\Omega) \subset L^2(\omega)$ we obtain

$$|L(\eta, \psi)| \leq 2\mu_1 \int_\Omega |D(v)|^2 |\psi| dx + C \int_\Omega |\psi| dx + \int_\omega |\theta_\omega| |\psi| dx',$$

and by Poincaré's and Hölder's inequalities, we obtain

$$|L(\eta, \psi)| \leq \left[2\mu_1 C_1 |\Omega|^{\frac{p-4}{2p}} \|D(v)\|_p^2 + C_2 |\Omega|^{\frac{1}{2}} + C_3 \|\theta_\omega\|_{L^2(\omega)} \right] \|\psi\|_{1,2} .$$

As $p \geq 4$, this proves the continuity of $L(\eta, .)$, and by Lax-Milgram Theorem, we deduce that there exists a unique $\theta \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$ solution of the linearized problem 3.3. $\square$

In the following theorem we prove only the existence of at least one solution to the intermediate Problem 3.2.

Theorem 3.4. *Let $\theta_\omega \in L^2(\omega)$ and $r \in L^\infty(\mathbb{R})$. Assume that functions r and $t \mapsto \mu(t,.,.)$ are Lipschitzian. Then with the same assumptions as* Theorem 3.3*, there exists at least one $\theta \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$ solution to the intermediate* Problem 3.2.

Proof. For $\eta \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$, Theorem 3.3 ensures the existence and the uniqueness of $\theta \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$ solution of the linearized problem (3.19). Then we can define the operator

$$\begin{array}{rcl} T : H^1_{\Gamma_1 \cup \Gamma_L}(\Omega) & \to & H^1_{\Gamma_1 \cup \Gamma_L}(\Omega) \\ \eta & \mapsto & T(\eta) = \theta, \end{array} \tag{3.21}$$

where θ is the unique solution of the linear problem 3.3.

We establish that T is completely continuous. For given η_1 (resp. η_2) in $H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$, we associate $T(\eta_1)$ (resp. $T(\eta_2)$), the solution of the equation (3.19). By substraction we obtain,

$$B\left(T(\eta_1) - T(\eta_2), \psi\right) = L(\eta_1, \psi) - L(\eta_2, \psi), \quad \forall \psi \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega). \tag{3.22}$$

By taking $\psi = T(\eta_1) - T(\eta_2)$ in (3.22) we get

$$\int_\Omega K(x) \left|\nabla \left(T(\eta_1) - T(\eta_2)\right)\right|^2 dx + P = Q + R, \tag{3.23}$$

where

$$\begin{array}{rcl} P & = & \displaystyle\int_\Omega \left(T(\eta_1) - T(\eta_2)\right) v_i \partial_i \left(T(\eta_1) - T(\eta_2)\right) dx, \\ Q & = & 2\displaystyle\int_\Omega \left[\mu\left(\eta_1, v, |D(v)|\right) - \mu\left(\eta_2, v, |D(v)|\right)\right] \left[T(\eta_1) - T(\eta_2)\right] |D(v)|^2 dx, \end{array}$$

and

$$R = \int_\Omega \left[r(\eta_1) - r(\eta_2)\right] \left[T(\eta_1) - T(\eta_2)\right] dx.$$

Let us evaluate the terms P, Q and R. By using the same arguments as in (3.20) we get

$$P = \frac{1}{2} \int_\Omega v_i \partial_i \left[\left(T(\eta_1) - T(\eta_2)\right)^2\right] dx = \frac{1}{2} \int_\Omega v_i \partial_i \left[(\theta_1 - \theta_2)^2\right] dx = 0. \tag{3.24}$$

For the trem Q we write,

$$|Q| \le 2C_\mu \int_\Omega |\eta_1 - \eta_2| \left|T(\eta_1) - T(\eta_2)\right| |D(v)|^2 dx, \tag{3.25}$$

where C_μ is the Lipschitz constant of the function $t \mapsto \mu(t,.,.)$.

Note here that if we took θ and η in $W^{1,q}_{\Gamma_1\cup\Gamma_L}(\Omega)$, the integrand in (3.25) would not be necessairily in $L^1(\Omega)$, this forced us to take $H^1_{\Gamma_1\cup\Gamma_L}(\Omega)$ instead of $W^{1,q}_{\Gamma_1\cup\Gamma_L}(\Omega)$.

Now we know from Rellich-Kondrachov Theorem that $H^1_{\Gamma_1\cup\Gamma_L}(\Omega)$ is compactly embedded in $L^4(\Omega)$. Then by taking η_i and $T(\eta_i)$ in $L^4(\Omega)$ we get

$$|Q| \le 2\,|\Omega|^{\frac{p-4}{2p}}\, C_\mu \left\|\eta_1-\eta_2\right\|_4 \left\|T(\eta_1)-T(\eta_2)\right\|_4 \left\|D(v)\right\|_p^2 .$$

Again by the compact embedding $H^1_{\Gamma_1\cup\Gamma_L}(\Omega) \subset L^4(\Omega)$, there exists a positive constant C' depending only on Ω such that

$$|Q| \le 2\,|\Omega|^{\frac{p-4}{2p}}\, C_\mu C' \left\|\eta_1-\eta_2\right\|_{1,2} \left\|T(\eta_1)-T(\eta_2)\right\|_{1,2} \left\|D(v)\right\|_p^2 . \tag{3.26}$$

Finally by Hölder's and Poincaré's inequalities we have

$$|R| \le C_r \left\|\eta_1-\eta_2\right\|_2 \left\|T(\eta_1)-T(\eta_2)\right\|_2 \le C_P C_r \left\|\eta_1-\eta_2\right\|_{1,2} \left\|T(\eta_1)-T(\eta_2)\right\|_{1,2} , \tag{3.27}$$

where C_P and C_r are respectively Poincaré's constant and Lipschitz's constant of the function r. Now by (3.17), (3.23), (3.24), (3.26) and (3.27) we obtain

$$k_0 \left\|T(\eta_1)-T(\eta_2)\right\|_{1,2}^2 \le \left(2\,|\Omega|^{\frac{p-4}{2p}}\, C_\mu C' \left\|D(v)\right\|_p^2 + C_P C_r\right) \left\|\eta_1-\eta_2\right\|_{1,2} \left\|T(\eta_1)-T(\eta_2)\right\|_{1,2} ,$$

and then

$$k_0 \left\|T(\eta_1)-T(\eta_2)\right\|_{1,2} \le \left(2\,|\Omega|^{\frac{p-4}{2p}}\, C_\mu C' \left\|D(v)\right\|_p^2 + C_P C_r\right) \left\|\eta_1-\eta_2\right\|_{1,2} .$$

This proves that the operator T is Lipschitzian. Let us now show that T is bounded in $H^1_{\Gamma_1\cup\Gamma_L}(\Omega)$. We know that for $\eta \in H^1_{\Gamma_1\cup\Gamma_L}(\Omega)$, $T(\eta)$ is solution of (3.19), that is

$$B(T(\eta),\psi) = L(\eta,\psi), \quad \forall \psi \in H^1_{\Gamma_1\cup\Gamma_L}(\Omega) .$$

By taking $\psi = T(\eta)$, and using (3.2) and (3.17) we obtain

$$k_0 \int_\Omega |\nabla T(\eta)|^2\, dx \le 2\mu_1 \int_\Omega |D(v)|^2\, |T(\eta)|\, dx + r_1 \int_\Omega |T(\eta)|\, dx + \int_\omega |\theta_\omega|\, |T(\eta)|\, dx,$$

where $r_1 = \text{ess sup}\,\{r(t), t \in \mathbb{R}\}$. From the continuous embedding $H^1_{\Gamma_1\cup\Gamma_L}(\Omega) \subset L^2(\omega)$, there exists a positive constant C'' independent of η such that

$$\left\|T(\eta)\right\|_{L^2(\omega)} \le C'' \left\|T(\eta)\right\|_{1,2} ,$$

and by Hölder's and Poincaré's inequalities, we get

$$\begin{aligned} k_0 \left\|T(\eta)\right\|_{1,2}^2 &\le 2\,|\Omega|^{\frac{p-4}{2p}}\, \mu_1 C_p \left\|D(v)\right\|_p^2 \left\|T(\eta)\right\|_{1,2} \\ &+ r_1 C_p \left\|T(\eta)\right\|_{1,2} + C'' \left\|\theta_\omega\right\|_{L^2(\omega)} \left\|T(\eta)\right\|_{1,2} . \end{aligned} \tag{3.28}$$

The boundedness of T follows from (3.28), $\|T(\eta)\|_{1,2} \leq C^*$, where

$$C^* = k_0^{-1}\left(2\,|\Omega|^{\frac{p-4}{2p}}\,\mu_1 C_p \,\|D(v)\|_p^2 + C''\,\|\theta_\omega\|_{L^2(\omega)} + C_p r_1\right).$$

Now according to Schauder Fixed Point Theorem we deduce that the operator T has at least one fixed point $\theta \in H^1_{\Gamma_1\cup\Gamma_L}(\Omega)$, solution of the variational problem (3.2). □

In the following theorem, we prove the uniqueness of the solution to the intermediate Problem 3.2.

Theorem 3.5. *Let $\theta_\omega \in L^2(\omega)$ and $r \in L^\infty(\mathbb{R})$. Assume that functions r and $t \mapsto \mu(t,.,.)$ are Lipschitzian and nonincreasing. So with the same assumptions as in* Theorem 3.3, *the solution of the intermediate* Problem 3.2 *is unique.*

Proof. Indeed, suppose in contrary that there exist two solutions θ_1 and θ_2 for (3.2). By substracting, we obtain for all $\psi \in H^1_{\Gamma_1\cup\Gamma_L}(\Omega)$,

$$\begin{aligned}\int_\Omega K(x)\nabla\Theta.\nabla\psi dx + \int_\Omega \psi v_i\partial_i\Theta dx &= 2\int_\Omega \left[\mu\left(\theta_1,v,|D(v)|\right) - \mu\left(\theta_2,v,|D(v)|\right)\right]|D(v)|^2\,\psi dx \\ &+ \int_\Omega \left(r(\theta_1) - r(\theta_2)\right)\psi dx, \qquad (3.29)\end{aligned}$$

where $\Theta = \theta_1 - \theta_2$. Now we use the real function f_δ (see eg [10,14]) defined for $\delta > 0$ by

$$f_\delta(t) = \begin{cases} (1-\frac{\delta}{t})^+ & if \quad t > 0,\\ 0 & if \quad t \leq 0,\end{cases}$$

with $A^+ = \max(A,0)$. As $\Theta \in H^1_{\Gamma_1\cup\Gamma_L}(\Omega)$ then $f_\delta(\Theta) \in H^1_{\Gamma_1\cup\Gamma_L}(\Omega)$ and

$$\nabla f_\delta(\Theta) = \frac{\delta}{\Theta^2}\chi_{[\Theta>\delta]}\nabla\Theta,$$

here $\chi_{[\Theta>\delta]}$ is the indicator function of the set $[\Theta > \delta] = \{x \in \Omega, \quad \Theta(x) > \delta\}$, that is

$$\chi_{[\Theta>\delta]}(x) = 1 \quad if \quad \Theta(x) > \delta, \quad \chi_{[\Theta>\delta]}(x) = 0 \quad if \quad \Theta(x) \leq \delta.$$

As $\psi = 0$ on $\Gamma_1 \cup \Gamma_L$, $v.n = 0$ on ω and $div(v) = 0$ in Ω, we have

$$\int_\Omega \psi v_i\partial_i\Theta dx = -\int_\Omega \Theta v.\nabla\psi dx. \qquad (3.30)$$

Taking $\psi = f_\delta(\Theta)$ in (3.29) and using (3.30) we get

$$\begin{aligned}\delta\int_{\Omega\cap[\Theta>\delta]} K(x)\left|\frac{\nabla\Theta}{\Theta}\right|^2 dx &= 2\int_{\Omega\cap[\Theta>\delta]} \frac{\mu\left(\theta_1,v,|D(v)|\right) - \mu\left(\theta_2,v,|D(v)|\right)}{\theta_1-\theta_2}\,|D(v)|^2\,(\Theta-\delta)\,dx \\ &+ \int_{\Omega\cap[\Theta>\delta]} \frac{r(\theta_1)-r(\theta_2)}{\theta_1-\theta_2}(\Theta-\delta)\,dx + \delta\int_{\Omega\cap[\Theta>\delta]} v\frac{\nabla\Theta}{\Theta}dx. \qquad (3.31)\end{aligned}$$

Since the functions r and $t \mapsto \mu(t,.,.)$ are nonincreasing then

$$\frac{\mu(\theta_1, v, |D(v)|) - \mu(\theta_2, v, |D(v)|)}{\theta_1 - \theta_2} \leq 0, \quad \text{and} \quad \frac{r(\theta_1) - r(\theta_2)}{\theta_1 - \theta_2} \leq 0. \tag{3.32}$$

Recall that $v \in V_{div}^p \subset (L^\infty(\Omega))^3$, so there exists a positive constant M independent of δ such that

$$\int_{\Omega \cap [\Theta > \delta]} v \frac{\nabla \Theta}{\Theta} dx \leq M \int_{\Omega \cap [\Theta > \delta]} \left| \frac{\nabla \Theta}{\Theta} \right| dx.$$

Now by (3.17), (3.32) and Cauchy-Schwarz inequality, (3.31) becomes

$$k_0 \int_{\Omega \cap [\Theta > \delta]} \left| \frac{\nabla \Theta}{\Theta} \right|^2 dx \leq M \int_{\Omega \cap [\Theta > \delta]} \left| \frac{\nabla \Theta}{\Theta} \right| dx \leq M |\Omega|^{\frac{1}{2}} \left(\int_{\Omega \cap [\Theta > \delta]} \left| \frac{\nabla \Theta}{\Theta} \right|^2 dx \right)^{\frac{1}{2}}.$$

Then

$$\begin{aligned} \left(\int_\Omega \left| \nabla \ln \left(1 + \frac{(\Theta - \delta)^+}{\delta} \right) \right|^2 dx \right)^{\frac{1}{2}} &= \left(\int_{\Omega \cap [\Theta > \delta]} \left| \frac{\nabla \Theta}{\Theta} \right|^2 dx \right)^{\frac{1}{2}} \\ &\leq M k_0^{-1} |\Omega|^{\frac{1}{2}}. \end{aligned} \tag{3.33}$$

The right hand side of (3.33) is independent of δ, then for $\delta \to 0$ we must obtain, $\Theta = \theta_1 - \theta_2 \leq 0$ a.e. in Ω, and by permuting the roles of θ_1 and θ_2 we get $\theta_2 - \theta_1 \leq 0$ then $\theta_1 = \theta_2$. This ends the proof of uniqueness of the temperature. □

3.3 Existence result for the coupled problem 2.1

We recall here the necessary assumptions to ensure the existence of at least one solution to the coupled problem 2.1.

We assume that, the real number $p \geq 4$, the function μ satisfies (3.1) and (3.2), the function K satisfies (3.17), the exterior force vector $f \in (W^{1,p}(\Omega))^3$, the upper limit for the stress $0 \leq k \in L^p(\omega)$, the given fixed flux θ_ω on ω is in $L^2(\omega)$, the real function $r \in L^\infty(\mathbb{R})$, we also suppose that functions r and $t \mapsto \mu(t,.,.)$ are Lipschitzian and nonincreasing.

Theorem 3.6. *Under the above assumptions, there exists a unique $\theta \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$ solution to* Problem 3.2 *and there exists at least one $(v_\theta, \pi_\theta) \in V_{div}^p \times L_0^p(\Omega)$ satisfying the variational inequality* (2.1).

Proof. For all $\eta \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega) \subset W^{1,q}_{\Gamma_1 \cup \Gamma_L}(\Omega)$, because $1 < q < 2$, by Theorem 3.1, there exists $v = v_\eta$ in V_{div}^p and by Theorem 3.2 there exists $\pi = \pi_\eta$ in $L_0^p(\Omega)$ solution to the variational inequality

$$a(\eta, v_\eta, \varphi - v_\eta) - (\pi_\eta, div(\varphi)) + j(\varphi) - j(v_\eta) \geq (f, \varphi - v_\eta) \qquad \forall \varphi \in V^q, \tag{3.34}$$

also by Theorems 3.4-3.5, there exists a unique $\theta \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$, solution to the Problem 3.2. So we can use the oprerator T defined by (3.21). By Theorem 3.4 we know that T has at least one fixed point $\theta \in H^1_{\Gamma_1 \cup \Gamma_L}(\Omega)$, $\theta = T(\theta)$, which is solution to the Problem 3.2. Then $(\theta, v_\theta, \pi_\theta)$ is solution to the Problem 2.1. $\square$

Remark 3.1. To our knowledge, the uniqueness of the problem 2.1 remains an open question.

Acknowledgments

The authors would like to thank the anonymous referee.

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